Long-term stochastic behavior of aeroelastic systems

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Abstract

An Adaptive Stochastic Finite Elements approach for unsteady problems is developed. Unsteady solutions of dynamical systems are known to be sensitive to small input variations. Stochastic Finite Elements methods usually require a fast increasing number of elements with time to capture the effect of random input parameters in these time-dependent problems. The resulting large number of samples required for resolving the long-term asymptotic stochastic behavior, results for computationally intensive fluid-structure interaction simulations in impractically high computational costs. The Unsteady Adaptive Stochastic Finite Elements (UASFE) formulation proposed in this paper maintains an approximately constant accuracy in time with a constant number of samples. The approach is based on a time-independent parametrization of the sampled time series in terms of frequency, phase, amplitude, reference value, damping, and higher-period shape function. This parametrization is interpolated using a robust Adaptive Stochastic Finite Elements method based on Newton-Cotes quadrature in simplex elements. The effectiveness of the UASFE approach is illustrated by applications to a mass-spring-damper system and the Duffing equation with random input parameters. The results are verified by comparison to those of Monte Carlo simulations.

1 Introduction

Mathematical models of engineering problems often contain input parameters which are subject to inherent physical randomness. If a system is sensitive to this variability, the random parameters can result in performance degeneration of a deterministically optimized design. The asymptotic behavior of dynamical systems with discontinuous solutions is known to be sensitive to small input variations. This type of problems is encountered in engineering practice in flutter analysis of the fluid-structure interaction of wing structures. Here, flutter refers to the loss of dynamical stability at a critical dynamic pressure to an oscillatory instability that can grow in an unbounded fashion [3]. Flutter of aeroelastic systems is of interest to engineers, since it can lead to structural failure and fatigue damage of the wing structure [8]. In these applications randomness can occur in, for example, material properties, fluid parameters, and boundary conditions, such as structural stiffness, structural damping, geometry, fluid viscosity, and atmospheric conditions.

It is recognized that predicting the large effect of random parameters in fluid-structure interaction problems is a challenging task. Performing Monte Carlo (MC) simulations [4], in which many deterministic problems are solved for randomly varying parameter values, is in this case not an economically feasible option, since deterministic fluid-structure interaction simulations are already computationally intensive. More efficient numerical methods are, therefore, needed which require in essence less deterministic solves to capture the effect of random input parameters.

Adaptive Stochastic Finite Elements (ASFE) methods [2, 5, 6, 10, 13] are examples of more efficient alternatives for Monte Carlo simulation. The Adaptive Stochastic Finite Elements (ASFE) approach based on Newton-Cotes quadrature in simplex elements [13] is employed in this paper, since it is a non-intrusive multi-element approach which requires a relatively low number of deterministic solves. The approach is based on a

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piecewise quadratic approximation of the response surface by dividing probability space into simplex elements. The approximation is constructed by collocating the problem in second-degree Newton-Cotes quadrature points in the elements. The elements are refined adaptively using a refinement measure based on the curvature of the response surface approximation weighted by the probability represented by the elements.

Successful applications of ASFE with Newton-Cotes quadrature and simplex elements are given in [13]. However, for resolving the time evolution of unsteady problems, Adaptive Stochastic Finite Elements methods usually require an increasing number of elements with increasing integration time [10]. This problem is especially profound in dynamical systems with oscillatory solutions in which the frequency of the response is affected by the random parameters [9, 11]. Resolving the long-term asymptotic stochastic behavior of these systems requires a high number of elements, due to the increasing phase differences with increasing time.

ASFE is in this paper combined with an extension of Probabilistic Collocation for Limit Cycle Oscillations (PCLCO) [12]. PCLCO is an efficient method for resolving the long-term stochastic effect of random parameters on period-1 oscillations. In PCLCO, Probabilistic Collocation [1, 7] is applied to a time-independent parametrization of periodic time series instead of to the time-dependent samples themselves. Due to the time-independent parametrization the accuracy of PCLCO is approximately independent of time for a constant number of samples. For period-1 oscillations a suitable parametrization consists of the frequency, relative phase, amplitude, a reference value and the normalized period. These functionals are interpolated in PCLCO using the global polynomial approximation of Probabilistic Collocation.

PCLCO has successfully been applied to a harmonic oscillator, an airfoil flutter model, and the flow around an elastically mounted cylinder in [12]. Although PCLCO results in a significant reduction of computational costs to resolve the stochastic behavior of these dynamical systems, it is subject to the following four limitations:

1. The global polynomial approximation of the time-independent functionals is only appropriate if these functionals are sufficiently smooth. Especially near bifurcation points, at which the response exhibits a singularity, this assumption is not valid.

2. The PCLCO formulation is limited to periodic limit cycle oscillation responses. In practical applications it is not trivial to ensure *a priori* that periodic solutions exist for the relevant parameter range.

3. PCLCO has only been applied to period-1 oscillations. Dynamical systems result in practice also in higher-period motions.

4. The application of PCLCO has been limited to random inputs in terms of a single random parameter. In practical applications the number of random parameters is usually larger than one.

In this paper, an Adaptive Stochastic Finite Elements approach for unsteady problems is developed based on a combination of the robust ASFE formulation with Newton-Cotes quadrature in simplex elements and an extension of the time-independent parametrization of PCLCO to damped and higher-period oscillations. The applicability of Unsteady Adaptive Stochastic Finite Elements (UASFE) extends beyond that of PCLCO with respect to the four limitations mentioned above:

1. The robustness of ASFE enables the application of the proposed approach to problems with bifurcations, in which the time-independent functionals are non-smooth.

2. The effect of positive and negative damping is resolved by the inclusion of a damping parameter in the time-independent parametrization, such that there is no need to ensure the existence of periodic solutions.

3. The proposed UASFE formulation includes an algorithm for parametrizing multi-period oscillations.

4. Applications to dynamical systems with inputs consisting of multiple random parameters are presented.

UASFE is applicable to problems with oscillatory solutions in which the functionals frequency, phase, amplitude, reference value, damping, and period shape are well defined in the asymptotic range.

The proposed approach is applied to a linear mass-spring-damper system and the nonlinear Duffing equation with random input parameters. The results are compared to those of Monte Carlo simulations. The mass-spring-damper system is considered to study the effect of positive and negative damping on the stochastic results. Input randomness is assumed in a spring stiffness parameter, damping parameter, and a combination of both. The effect of random initial conditions is studied for the Duffing equation. The bifurcation behavior of the Duffing system results in a long-term solution which is highly sensitive to variations in the initial conditions. The current applications are limited to the asymptotic behavior of single-frequency rigid-body motions. Extension beyond these limitations needs further attention.

The paper is organized as follows. Unsteady Adaptive Stochastic Finite Elements are introduced in section 2. Numerical results for the mass-spring-damper system and the Duffing equation subject to random parameters are presented in section 3. The paper is concluded in section 4.

2 Unsteady Adaptive Stochastic Finite Elements

In this section Unsteady Adaptive Stochastic Finite Elements (UASFE) based on Newton-Cotes quadrature in simplex elements are developed. The employed Adaptive Stochastic Finite Elements (ASFE) framework is discussed in section 2.1. In section 2.2 the formulation for unsteady problems is introduced.

2.1 Adaptive Stochastic Finite Elements

Consider the following dynamical system

$$\mathcal{L}(\mathbf{x}, t; u(\mathbf{x}, t, \omega)) = S(\mathbf{x}, t), \tag{1}$$

subject to random input parameters, with operator $\mathcal{L}$ and source term S defined on domain $D \subset \mathbb{R}^d \times T$, $d = \{1, 2, 3\}, T = [0, t_{\mathrm{stop}}]$, and appropriate initial conditions. Equation (1) describes the system output response $u(\mathbf{x}, t, \omega)$, where $\mathbf{x} \in D$ and $t \in T$ denote the spatial and temporal dimensions, and $\omega \in \Omega$ is a realization of the set of outcomes Ω of the probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F} \subset 2^{\Omega}$ the σ-algebra of events and P a probability measure. The dimension n of probability space Ω is equal to the number of independent second-order random input parameters $\mathbf{a}(\omega) = \{a_1(\omega), \ldots, a_n(\omega)\}$, with $\mathbf{a}(\omega) \in A$. A two-dimensional probability space with $n = 2$ random input parameters $\mathbf{a}(\omega) = \{a_1(\omega), a_2(\omega)\}$ is considered in this paper.

Adaptive Stochastic Finite Elements based on Newton-Cotes quadrature in simplex elements [13] divide parameter space A into N_{e} simplex elements A_i for $i = 1, \ldots, N_{\mathrm{e}}$. A piecewise quadratic approximation of $u(\mathbf{x}, t, \omega)$ is constructed based on $\tilde{N}_{\mathrm{s}} = 6$ deterministic solves in the second-degree Newton-Cotes quadrature points in the elements A_i, see Figure 1a. The m^{th} statistical moment $\mathbb{E}(u(\mathbf{x}, t, \omega)^m)$ of $u(\mathbf{x}, t, \omega)$ is then approximated by

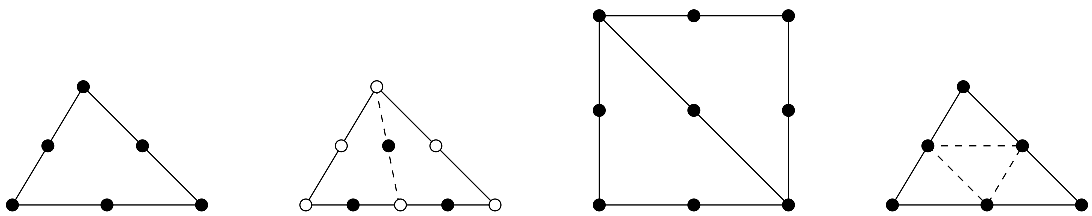
$$\mathbb{E}(u(\mathbf{x}, t, \omega)^m) = \int_{\Omega} u(\mathbf{x}, t, \omega)^m \mathrm{d}\omega \approx \sum_{i=1}^{N_{\mathrm{e}}} \sum_{j=1}^{\tilde{N}_{\mathrm{s}}} c_{i,j} u_{i,j}(\mathbf{x}, t)^m, \tag{2}$$

where $c_{i,j}$ are Newton-Cotes quadrature weights and $u_{i,j}(\mathbf{x}, t)$ are realizations of the response $u(\mathbf{x}, t, \omega)$ for the parameter values $\mathbf{a}_{i,j}$ in the quadrature points in element A_i. The quadrature weights $c_{i,j}$ are given by the second-degree Newton-Cotes formula in parameter space A weighted by the probability distribution of the random input parameters $\mathbf{a}(\omega)$.

The elements A_i are adaptively refined using refinement measure ρ_i based on the curvature of the response surface approximation in the elements weighted by the probability represented by the elements. As measure for the curvature the largest absolute eigenvalue of the Hessian of the quadratic approximation in the elements is used. In the refinement the element A_i with the highest value of refinement measure ρ_i is refined into two simplex elements. The longest edge of the element is split into two halves of equal length and maximal 3 new samples are computed, which are given in Figure 1b by the black dots. The new refinement measures in the refined element are determined and the element with the largest refinement measure is again refined, etc. The initial grid consists of $N_{\mathrm{e_{ini}}} = 2$ elements and $N_{\mathrm{s_{ini}}} = 9$ samples, see Figure 1c.

Due to the Newton-Cotes quadrature the required number of deterministic solves is relatively low, since (1) the deterministic samples are reused in successive refinement steps due to the location of the quadrature points, and (2) the samples are used in approximating the response in multiple elements, because most quadrature points are located on the boundaries of the elements. This results for uniform grid refinement in an average number of samples per element $N_{\mathrm{s}}/N_{\Omega}$ that approaches 2.

In order to preserve monotonicity and optima of the samples in the piecewise polynomial approximation, the elements are subdivided into 4 simplex subelements with a linear trapezoidal rule approximation of the response where necessary, see Figure 1d. An element is split into subelements when the quadratic approximation of the



(a) Element (b) Refinement (c) Initial grid (d) Subelements

Figure 1: The 2-simplex element A_i in parameter space A and the second-degree Newton-Cotes quadrature points given by the dots.

response in the element has an optimum other than in a quadrature point. This prevents unphysical predictions due to over- and undershoots near singularities.

2.2 Adaptive Stochastic Finite Elements for unsteady problems

In unsteady problems, Adaptive Stochastic Finite Elements methods can result in a fast increasing number of elements with time. The Unsteady Adaptive Stochastic Finite Elements (UASFE) formulation proposed here is based on a time-independent parametrization of the sampled time series to enable a constant accuracy in time with a constant number of samples. The time-independent parametrization developed below is an extension of the parametrization employed in Probabilistic Collocation for Limit Cycle Oscillations (PCLCO) [12] to damped and multi-period oscillations. For the parametrization, $u(t,\omega)$ is approximated by the following representation $\tilde{u}(t,\omega)$:

$$\tilde{u}(t,\omega) = u_0(\omega) + \mathrm{e}^{\gamma(\omega)(t_{\mathrm{stop}}-t)}A(\omega)u_{\mathrm{period}}(\tau(\omega),\omega), \tag{3}$$

and $\tau(\omega) = 2\pi(\phi(\omega) + (t - t_{\mathrm{stop}})f(\omega)) \pmod{2\pi}$, with t_{stop} the end time of the numerical time integration. The argument $\mathbf{x}$ has been dropped here for clarity. The response $u(t,\omega)$ is parametrized by (3) in terms of the frequency $f(\omega)$, relative phase $\phi(\omega)$, amplitude $A(\omega)$, a reference value $u_0(\omega)$, damping $\gamma(\omega)$, and normalized period $u_{\mathrm{period}}(\tau,\omega)$, with $\tau \in [0, 2\pi]$, see Figure 2.

The parametrization is extracted from the N_{s} sampled solutions $u_k(t)$ of (1) for the parameter values $\mathbf{a}(\omega)$ of the Newton-Cotes quadrature points in the elements. The N_{s} deterministic time series $u_k(t)$ result in N_{s} realizations of f_k, ϕ_k, A_k, u_{0_k}, γ_k, and $u_{\mathrm{period}_k}(\tau)$. ASFE are used to interpolate these realizations, for example f_k for $k = 1,\dots,N_{\mathrm{s}}$, to the functional $f(\omega)$. The functionals are substituted into (3) to obtain an approximation of $u(t,\omega)$. The Unsteady Adaptive Stochastic Finite Elements algorithm consists of the following 10 steps:

1. Perform the deterministic solves of the initial grid;
2. Extract the local optima from the time series;
3. Select the last completed period of the time series;
4. Extract the time-independent parametrization from the last completed periods;
5. Determine the parametrization error in the reconstruction of the samples;
6. Detect higher-period oscillations in the time series;
7. Determine the valid time interval of all samples;
8. Construct the response approximation on the initial grid;
9. Refine the stochastic grid and repeat steps 2 to 8 for the new samples;
10. Stop the refinement based on convergence in the L_∞-norm.

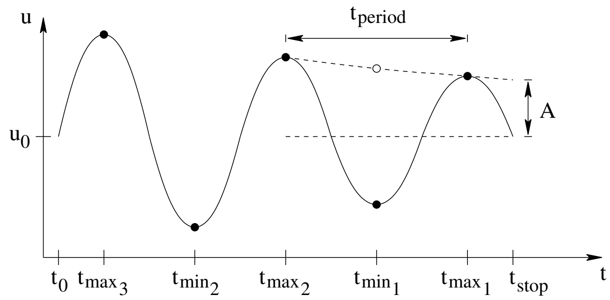


Figure 2: Time-independent parametrization of oscillatory time series employed in Unsteady Adaptive Stochastic Finite Elements.

3 Results

The Unsteady Adaptive Stochastic Finite Elements (UASFE) formulation is applied to a mass-spring-damper system and the Duffing equation in sections 3.1 and 3.2, respectively. The results are compared to those of Monte Carlo (MC) simulations. The number of samples in the Monte Carlo simulations is established after performing convergence studies.

3.1 Mass-spring-damper system

A mass-spring-damper system of a mass attached to a spring and a damper is considered with randomness in the spring stiffness parameter $K(\omega)$ and the damping constant $C(\omega)$. The governing equation for the motion of the mass is given in section 3.1.1. The combined effect of randomness in $K(\omega)$ and $C(\omega)$ is studied in section 3.1.2.

3.1.1 Governing mass-spring-damper equation

Consider a mass attached to a spring and a damper as shown in Figure 3. This can be a model for a more complex structure with internal damping and stiffness. The mass-spring-damper system is governed by

$$M\frac{\partial^2 x}{\partial t^2} + C(\omega)\frac{\partial x}{\partial t} + K(\omega)x = 0, \qquad t \in [0, \infty), \tag{4}$$

with mass $M = 1$, position of the mass $x(t, \omega)$, and initial conditions $x(0) = 1$ and $\partial x/\partial t(0) = 1$. The randomness in the positive spring stiffness $K(\omega)$ is given by a lognormal distribution with mean $\mu_{\mathrm{K}} = 1$ and coefficient of variation $\mathrm{COV_K} = 10\%$. A normal distribution is assumed for $C(\omega)$ with mean $\mu_{\mathrm{C}} = 0$ and standard deviation $\sigma_{\mathrm{C}} = 0.01$. As is common in multi-element methods, the tails of the probability distributions for $K(\omega)$ and $C(\omega)$ are truncated such that the resolved parameter domains account for 99.8% of the realizations. The resulting truncation error is small compared to the usual discretization and time integration errors in solving computational engineering problems. The results below are based on the analytical solution of (4)

$$x(t, \omega) = c_1 e^{c_2 t}\sin(c_3 t) + c_4 e^{c_5 t}\cos(c_6 t), \tag{5}$$

where c_i with $i = 1, \ldots, 6$ are functions of M, $K(\omega)$, $C(\omega)$, $x(0)$, and $\partial x/\partial t(0)$. Eq. (5) is evaluated at discrete time levels $t_l = l\Delta t$, with $l = 0, \ldots, n_{\mathrm{t}}$, $n_{\mathrm{t}} = t_{\mathrm{stop}}/\Delta t$, $t_{\mathrm{stop}} = 100$, and $\Delta t = 0.01$, for the results to be comparable with the results of the other test problem obtained by numerical time integration. The time interval corresponds to approximately 16 periods for the deterministic case with $K = \mu_{\mathrm{K}}$ and $C = \mu_{\mathrm{C}}$.

3.1.2 Random $K(\omega)$ and $C(\omega)$

The combined effect of randomness in both the spring stiffness parameter $K(\omega)$ and the damping $C(\omega)$ is considered. To this end, a two-dimensional UASFE formulation is employed to discretize the two-dimensional probability space. The approximations of the response surface of $x(t, \omega)$ at t_{stop} as function of the random parameters $K(\omega)$ and $C(\omega)$ by UASFE and Monte Carlo simulation are given in Figure 4. The UASFE approximation on

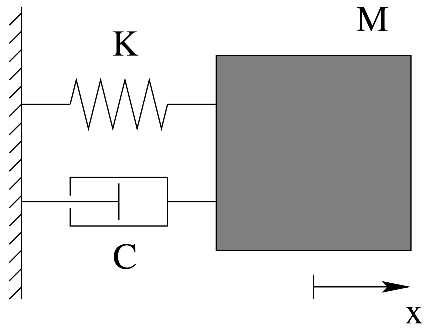
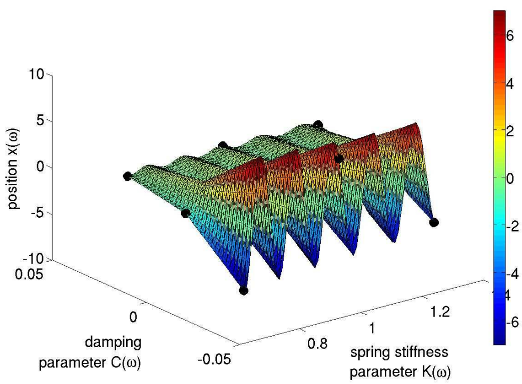
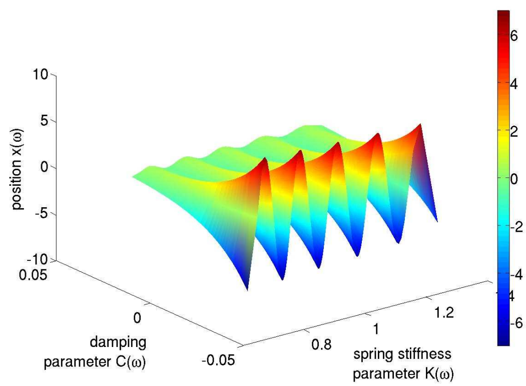


Figure 3: The mass-spring-damper system.



(a) UASFE, $N_{\rm s} = 9$ samples



(b) Monte Carlo, $N_{\rm s} = 1.2 \cdot 10^5$ samples

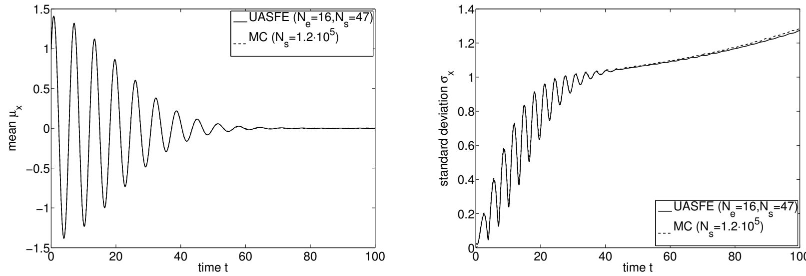
Figure 4: Two-dimensional response surface of $x(t,\omega)$ at $t_{\rm stop} = 100$ as function of the random stiffness $K(\omega)$ and damping $C(\omega)$ by two-dimensional Unsteady Adaptive Stochastic Finite Elements (UASFE) with $N_{\rm e} = 2$ ($N_{\rm e_{sub}} = 4^6$, $N_{\rm s} = 9$) and Monte Carlo (MC) simulation with $N_{\rm s} = 1.2\cdot 10^5$ for the mass-spring-damper system.

the initial grid of $N_{\rm e_{ini}} = 2$ elements and $N_{\rm s_{ini}} = 9$ samples is shown and the Monte Carlo simulation result for $N_{\rm s} = 1.2\cdot 10^5$ samples is considered. The response surface shows an oscillatory behavior in the $K(\omega)$-dimension. In the $C(\omega)$-dimension the initial deflection of $x_0(0) = 1$ is damped for $C(\omega) > 0$ and amplified for $C(\omega) < 0$. Unsteady Adaptive Stochastic Finite Elements capture this complex nonlinear behavior already with $N_{\rm s_{ini}} = 9$ samples of the initial grid. In the post-processing of UASFE, $N_{\rm e_{sub}} = 4^6$ subelements are employed per element for the result of Figure 4 without performing additional deterministic solves.

The results of UASFE and Monte Carlo simulation for the mean $\mu_{\rm x}(t)$ and standard deviation $\sigma_{\rm x}(t)$ are shown in Figure 5. UASFE converge for $N_{\rm e} = 16$ elements and $N_{\rm s} = 47$ samples below $5 \cdot 10^{-2}$ in the L_∞-norm. The post-processing converges for $N_{\rm e_{sub}} = 4^4$ subelements per element. The mean $\mu_{\rm x}(t)$ is a decaying oscillation and the standard deviation $\sigma_{\rm x}(t)$ has initially an oscillatory behavior until approximately $t = 50$ due to the random spring stiffness $K(\omega)$. For $t > 50$, $\sigma_{\rm x}(t)$ shows a monotonically increasing behavior due to the non-zero probability of negative values for the damping $C(\omega)$.

3.2 Duffing equation

In this section the effect of random initial conditions for the Duffing differential equation is studied. The Duffing equation is a model for a damped oscillator with a cubic nonlinear spring. Parameter variations are selected for which the response shows a discontinuous change to a qualitatively different behavior. The Duffing differential



(a) mean μ_{x} (b) standard deviation σ_{x}

Figure 5: Mean $\mu_{\mathrm{x}}(t)$ and standard deviation $\sigma_{\mathrm{x}}(t)$ by two-dimensional Unsteady Adaptive Stochastic Finite Elements (UASFE) with $N_{\mathrm{e}} = 16$ ($N_{\mathrm{s}} = 47$) and Monte Carlo (MC) simulation with $N_{\mathrm{s}} = 1.2 \cdot 10^5$ for the mass-spring-damper system with random spring stiffness $K(\omega)$ and damping $C(\omega)$.

equation is described in section 3.2.1. The effect of two random initial conditions is considered in section 3.2.2.

3.2.1 Duffing system of differential equations

The Duffing system of equations is given by the following differential equations:

$$\frac{\partial x}{\partial t} = y \tag{6}$$

$$\frac{\partial y}{\partial t} = \omega_0^2 x - \beta x^3 - \delta y + \gamma \cos(\omega_\gamma t + \phi), \tag{7}$$

for $t \in [0, \infty)$ with initial conditions $x(0) = x_0$ and $y(0) = y_0$. The Duffing equation is a model for a damped oscillator with damping δ and a cubic nonlinear spring stiffness term with parameter β. According to (6), $x(t)$ can be interpreted as a deflection of a mass with velocity $y(t)$. The acceleration $\partial y/\partial t$ is then governed by (7). A hard spring with increasing spring stiffness as function of deflection $x(t)$ is modeled by $\beta > 0$ and $\beta < 0$ holds for a soft spring. Structural stiffness behaves as a cubic hard spring in, for example, the torsional direction of wing structures [14]. Here, the cubic spring stiffness parameter is assumed to be $\beta = 1$. The undamped and unforced Duffing equation is considered, i.e. $\delta = 0$ and $\gamma = 0$, and ω_0 is chosen to be unity.

The resulting dynamical system has fixed points (x, y) in $(-1, 0)$, $(0, 0)$, and $(1, 0)$. The fixed point $(0, 0)$ is unstable, and $(-1, 0)$ and $(1, 0)$ are linearly stable, see the x–y phase diagram in Figure 6. The solution exhibits a periodic trajectory around either the linearly stable points $(-1, 0)$ or $(1, 0)$, or around all three fixed points. Which type of system response is found, depends on the initial conditions x_0 and y_0. Since the qualitative behavior of the solution is sensitive to the initial conditions, randomness is considered in $x_0(\omega)$ and $y_0(\omega)$. The system (6) and (7) is solved numerically up to $t_{\mathrm{stop}} = 100$ using fourth-order explicit Runge-Kutta time integration with a time step of $t = 0.01$.

3.2.2 Random $x_0(\omega)$ and $y_0(\omega)$

Assuming randomness in both initial conditions $x_0(\omega)$ and $y_0(\omega)$ results in a two-dimensional probability space, to which two-dimensional Unsteady Adaptive Stochastic Finite Elements are applied. The initial conditions $x_0(\omega)$ and $y_0(\omega)$ are assumed to be uniformly distributed in the interval $[0.45; 0.55]$. This interval corresponds to a mean of $\mu_{\mathrm{x}_0} = \mu_{\mathrm{y}_0} = 0.5$ and standard deviation of $\sigma_{\mathrm{x}_0} = \sigma_{\mathrm{y}_0} = 8.33 \cdot 10^{-4}$. This parameter domain is chosen since it contains a bifurcation of the Duffing equation. Since the time series are periodic solutions, the UASFE reconstruction is valid for all $t \in [0, t_{\mathrm{stop}}]$. The presented results are converged below $5 \cdot 10^{-2}$ in the L_∞-norm.

The bifurcation of the solution is illustrated by the two-dimensional response surfaces of $x(t, \omega)$ and $y(t, \omega)$ at $t = t_{\mathrm{stop}}$ as function of the random initial conditions $x_0(\omega)$ and $y_0(\omega)$ shown in Figure 7. In Figures 7a and 7b the approximation of the response surface of $x(t, \omega)$ by UASFE with $N_{\mathrm{e}} = 64$ elements ($N_{\mathrm{e_{sub}}} = 4^4$)

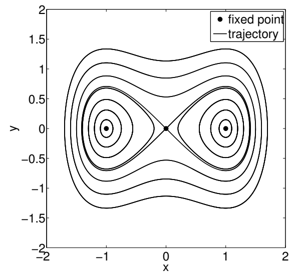


Figure 6: Phase diagram for the Duffing equation in terms of fixed points $(\pm 1, 0)$ and $(0, 0)$, and trajectories for $\beta = 1$, $\delta = 0$, $\gamma = 0$, and $\omega_0 = 1$.

and $N_\mathrm{s} = 151$ samples, and Monte Carlo with $N_\mathrm{s} = 10^4$ samples are given, respectively. For $y(t, \omega)$ the corresponding results are shown in Figures 7c and 7d. The results of UASFE agree well with the Monte Carlo results at a reduction of the number of samples by more than a factor 50. The response surfaces for $x(t, \omega)$ and $y(t, \omega)$ are highly oscillatory due to the alternating positive and negative values of both $x(t, \omega)$ and $y(t, \omega)$. In the response of $x(t, \omega)$ a bifurcation can be recognized at the line between (x_0, y_0) equal to $(0.475; 0.45)$ and $(0.55; 0.5)$, see Figures 7a and 7b. Below this line, only positive values of $x(t, \omega)$ are found, which corresponds to periodic trajectories around fixed point $(1, 0)$. Positive and negative values of $x(t, \omega)$ occur above the bifurcation line, which corresponds to periodic solutions around all three fixed points. In the response surface of $y(t, \omega)$ the bifurcation can be identified at the same location.

UASFE capture the discontinuous bifurcation behavior by refining the elements near the bifurcation line. In Figure 8 the grid of UASFE with $N_\mathrm{e} = 64$ elements in probability space is given. The $N_\mathrm{s} = 151$ samples are denoted by the dots. The elements are clearly more refined near the bifurcation than in the continuous domains. The smallest elements in the grid are 16 times smaller than the largest ones.

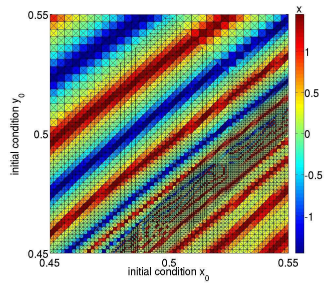
In Figure 9 it is demonstrated that UASFE agrees with the Monte Carlo results also for the mean and the standard deviation of $x(t, \omega)$ and $y(t, \omega)$ for $t \in [0, t_\mathrm{stop}]$. The mean $\mu_\mathrm{x}(t)$ of $x(t, \omega)$ is a damped oscillation with a positive asymptotic value of $\mu_\mathrm{x} = 0.15$, see Figure 9a. The mean $\mu_\mathrm{y}(t)$ of $y(t, \omega)$ is an oscillation which decays to zero. Both standard deviations $\sigma_\mathrm{x}(t)$ and $\sigma_\mathrm{y}(t)$ are irregular oscillations which approach finite asymptotic values of $\sigma_\mathrm{x}(t) = 0.75$ and $\sigma_\mathrm{y}(t) = 0.46$. The results show an amplification of the standard deviation of the random initial conditions by factors $9.0 \cdot 10^2$ and $5.5 \cdot 10^2$ for $x_0(\omega)$ and $y_0(\omega)$, respectively.

4 Conclusions

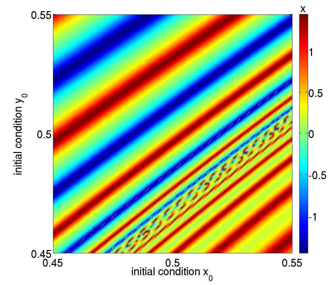
An Unsteady Adaptive Stochastic Finite Elements (UASFE) formulation for unsteady problems is developed based on a time-independent parametrization of deterministic time series. Due to the time-independent parametrization, UASFE maintain an approximately constant accuracy in time with a constant number of elements. The parametrization of the samples consists of the frequency, phase, amplitude, reference value, damping, and period shape. The parameters are interpolated using a robust Adaptive Stochastic Finite Element (ASFE) method based on Newton-Cotes quadrature in simplex elements. This approach requires a relatively low number of deterministic solves and preserves monotonicity and optima of the samples. In order to ensure the robustness of the method, (1) the elements are refined adaptively until convergence is reached in the L_∞-norm, and (2) the parametrization error is computed to determine the time interval in which the UASFE approximation is valid.

The robustness of the ASFE interpolation enables the application of the proposed approach to problems with bifurcations, in which the time-independent functionals are non-smooth. The effect of positive and negative damping is resolved by the inclusion of a damping parameter in the time-independent parametrization. The UASFE formulation includes an algorithm for parametrizing multi-period oscillations.

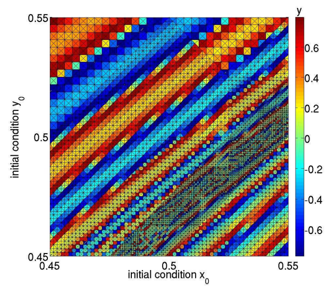
Results for a mass-spring-damper system and the Duffing equation with multiple random input parameters are presented. For the mass-spring-damper system the effect of positive and negative damping on the stochastic results is studied. Input randomness assumed in a combination of the spring stiffness parameter and the damping parameter shows that a non-zero probability of negative damping results asymptotically in a diverging output



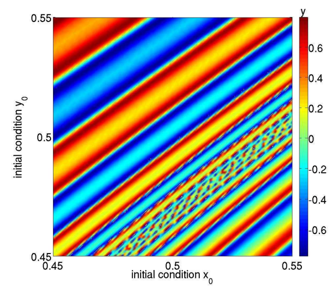
(a) UASFE $x(t, \omega)$



(b) MC $x(t, \omega)$



(c) UASFE $y(t, \omega)$



(d) MC $y(t, \omega)$

Figure 7: Two-dimensional response surface approximations for $x(t_{\text{stop}}, \omega)$ and $y(t_{\text{stop}}, \omega)$ as function of random initial conditions $x_0(\omega)$ and $y_0(\omega)$ by Unsteady Adaptive Stochastic Finite Elements (UASFE) with $N_{\text{e}} = 64$ ($N_{\text{e}_{\text{sub}}} = 4^4$, $N_{\text{s}} = 151$) and Monte Carlo (MC) with $N_{\text{s}} = 10^4$ for the Duffing equation.

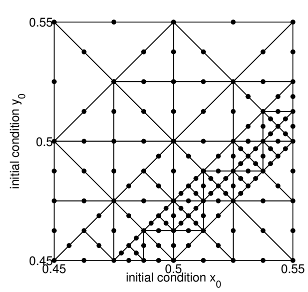


Figure 8: Unsteady Adaptive Stochastic Finite Elements (UASFE) grid in two-dimensional parameter space $x_0(\omega)$–$y_0(\omega)$ with $N_{\text{e}} = 64$ ($N_{\text{s}} = 151$) given by the dots for the Duffing equation.

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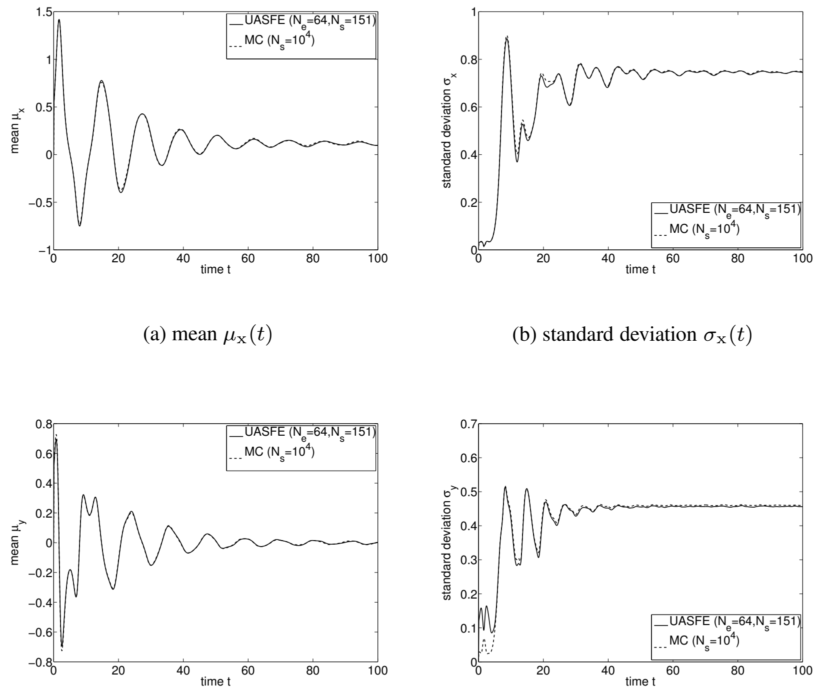

(a) mean $\mu_{\mathrm{x}}(t)$

(b) standard deviation $\sigma_{\mathrm{x}}(t)$

(c) mean $\mu_{\mathrm{y}}(t)$

(d) standard deviation $\sigma_{\mathrm{y}}(t)$

Figure 9: Mean $\mu_{\mathrm{x}}(t)$ and $\mu_{\mathrm{y}}(t)$, and standard deviation $\sigma_{\mathrm{x}}(t)$ and $\sigma_{\mathrm{y}}(t)$ by Unsteady Adaptive Stochastic Finite Elements (UASFE) with $N_{\mathrm{e}} = 64$ ($N_{\mathrm{s}} = 151$) and Monte Carlo (MC) with $N_{\mathrm{s}} = 10^4$ for the Duffing equation with random initial conditions $x_0(\omega)$ and $y_0(\omega)$.

standard deviation. The study of two random initial conditions for the Duffing equation illustrates that nonlinear dynamical systems with discontinuous solutions can be extremely sensitive to random initial conditions. An amplification factor of $9.0 \cdot 10^2$ has been observed for the standard deviation.

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